MMP Learning Seminar

Week 2:

Bend and break,
rational curves,
Cone Theorem

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Week 2:

1.- Bend and break,
2.- Finding rat curves when K_X is not nef,
3.- The Cone Theorem.

1. Bend and break:

(B&B I)

Proposition: X proper, C smooth proper curve $p \in C$, $g_0 : C \to X$ non-const.

$0 \in D$ pointed curve, $G : C \times D \to X$ s.t

(1) $G|_{C \times \{0\}} = g_0$,

(2) $G(\{p\} \times D) = g_0(p)$ and

(3) $G|_{C \times \{t\}}$ is diff then from g_0 for general t.

There exists $g_1 : C \to X$, $Z = \sum_{a_i > 0} a_i Z_i$ of rat curves so that

1) $(g_0)_* C \sim_{alg} (g_1)_* (C) + Z$, and.

2) $g_0(p) \in \bigcup_i Z_i$.

In particular there is a rat curve through $g_0(p)$.

Proof: $\bar{G} : C \times \bar{D} \dashrightarrow X$, is undefined at $\{p\} \times \bar{D}$ (Rigidity Lemma).

S the norm of the graph of $\bar{G}$, $\pi : S \to C \times \bar{D}$, $G_S : S \to X$.

$h : S \to C \times \bar{D} \to \bar{D}$.

There exists $(p, d) \in C \times \bar{D}$ so that π is not an isom over (p, d)

$h^{-1}(d) = C' + E$, C' bir transform of C, E π-exc.

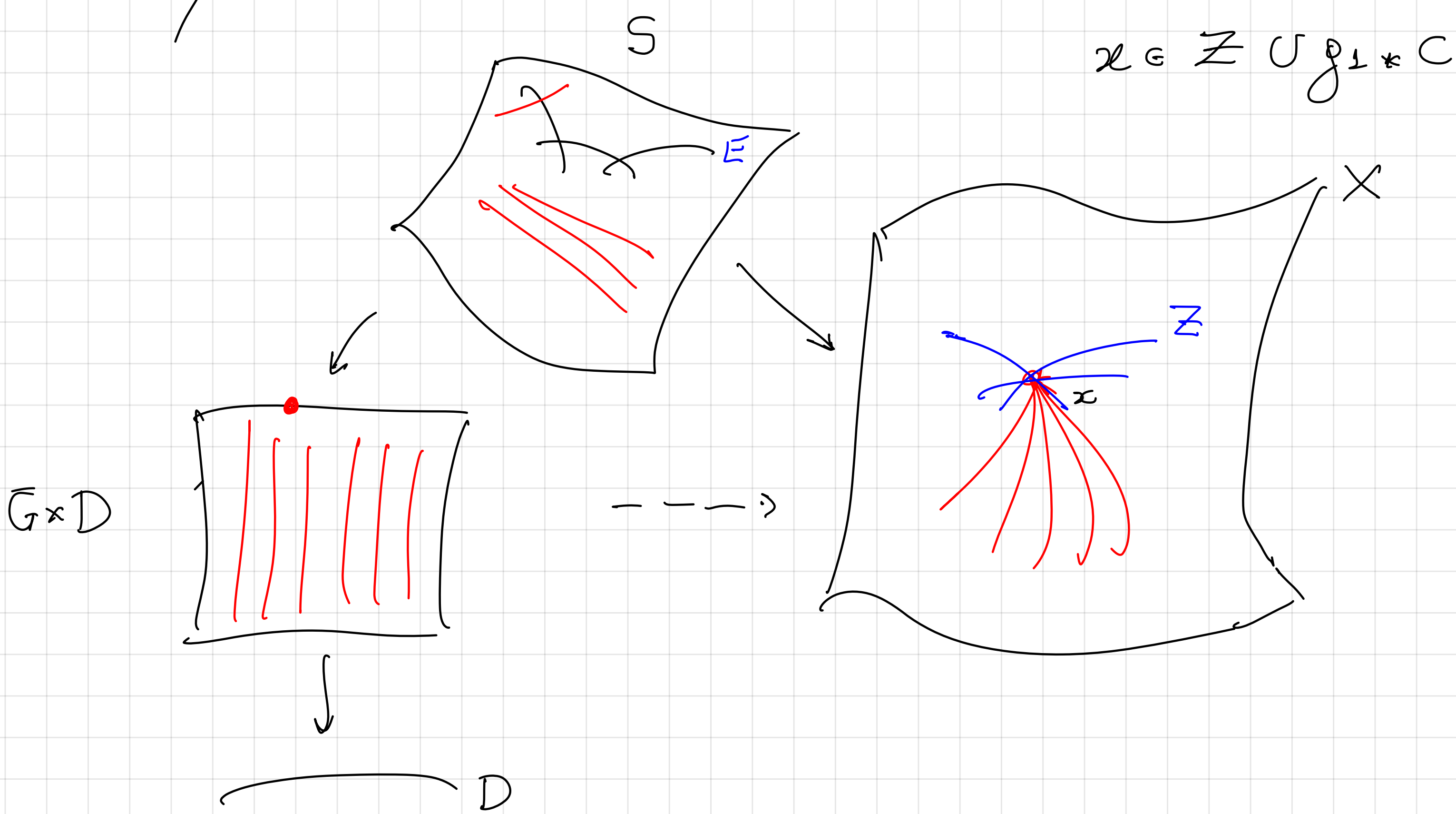
$g_1 : C \to X$, restriction of G_S to C' and $Z = G_S(E)$.

(Abhyankar Lemma): E is a union of rat curves.

(Luroth Thm): Z is a union of rat curves.

$$(g_0)_* C \sim_{alg} (g_1)_* C + Z.$$

Abhyankar Lemma: X has smooth mild sing and $Y \xrightarrow{\pi} X$ proper birational morphism. For any $x \in X$, either $\pi^{-1}(x)$ is a point or is covered by rat curves.



(B&B II).

Proposition: Let X be a proj var, $g_0 : \mathbb{P}^1 \to X$ non-const.

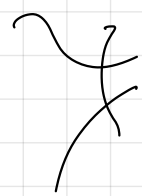
D smooth point curve. $G : \mathbb{P}^1 \times D \longrightarrow X$ st:

1) $G|_{\mathbb{P}^1 \times \{0_D\}} = g_0$,

2) $G(\{0\} \times D) = g_0(0)$, $G(\{\infty\} \times D) = g_0(\infty)$, and

3) $G(\mathbb{P}^1 \times D)$ is a surface

Then $(g_0)_* \mathbb{P}^1$ is $\sim_{alg}$ to a reducible curve or a multiple curve



Proof: $\bar{G} : S \dashrightarrow X$, S is a $\mathbb{P}^1$-bundle containg $\mathbb{P}^1 \times D$.

$\tilde{G} : \tilde{S} \longrightarrow X$, do induction on $\rho(\tilde{S}/S) = \rho$

$\tilde{S} \longrightarrow S$

Case 1: $\rho = 0$, C_0 and C_∞ two sections at $\{0\}$ and $\{\infty\}$

H ample on X, $(\tilde{G}^*H)^2 > 0$ and $(C_0 \cdot \tilde{G}^*H) = (C_\infty \cdot \tilde{G}^*H) = 0$.

$\boxed{\begin{array}{l} \textbf{Proj formula:} \quad f: Y \longrightarrow X \\ C \subseteq Y \qquad\qquad \mathcal{L} \text{ line bundle} \\ f^*\mathcal{L} \cdot C = \mathcal{L} \cdot f_* C \end{array}}$

$C_0^2 < 0$, $C_\infty^2 < 0$

$C_0 \cdot C_\infty = 0$

$\boxed{\begin{array}{l} \text{Hodge index Thm: if } H^2 > 0 \text{ for some curve.} \\ \text{then the self int form} \\ \text{is neg def in } H^\perp. \end{array}}$

$\tilde{G}^*H$, C_0 and C_∞ are l.i. $\qquad a\tilde{G}^*H + bC_0 + cC_\infty = 0$.

$\rho(S) = 2$. $\qquad \longrightarrow \longleftarrow$

Case 2:

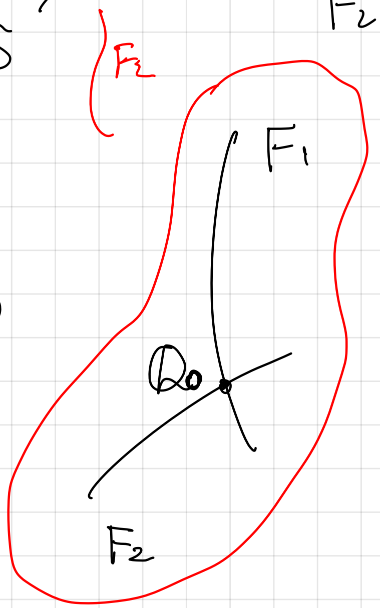
$$\begin{tikzcd}
\tilde S \arrow[r, "\tilde G"] \arrow[d, "r'"'] \arrow[dd, bend right=60, "\tilde r"'] & X \\
S' \arrow[ur, dashed, "\bar G' \; ?"] \arrow[d, "r"'] & \\
S \arrow[uur, dashed, "\bar G"'] \arrow[d, "q"'] & \\
D &
\end{tikzcd}$$

r is the first blow-up in $\tilde S \to S \ni P$

$y \in D$ will be the point s.t. $q^{-1}(y) \ni P$

F_1 is the exc of r.

F_2 is the strict tran of $q^{-1}(y)$ in S'.



Claim: $\bar G'$ is a morphism around F_2

$$g_0 * \mathbb{P}^1 \sim \tilde G_*((q \circ r)^*(y))$$

irreducible + red.

Assume $\bar G$ is not defined at $Q \neq P$.

$$\tilde G_*((q\circ r)^*(y)) = \tilde G_* \operatorname{red}(\tilde r^{-1}(P)) + \tilde G_* \operatorname{red}(\tilde r^{-1}(Q)) + (\text{eff}) \longrightarrow \longleftarrow$$

Assume $\bar G'$ is not defined at Q_0, in this case we need to blow-up Q_0.

so $(q \circ r)^*(y)$ contains a comp of mult ≥ 2.

Theorem: X smooth proj, $-K_X$ ample. For every $x \in X$, there exists a rat curve C through x s.t.

$$0 < -K_X \cdot C \leq \dim X + 1$$

Proof: Pick $C \subseteq X$ through x.

The space of def of C on X fixing x has dim $\geq$

$$h^0(C, f^*T_X) - h^1(C, f^*T_X) = -f_*C \cdot K_X - g(C)\dim X.$$
$$- \dim X$$

(1) $g(C) = 0$, ✓

(2) $g(C) = 1$, $C \xrightarrow{h} C$

$$-((f \circ h)_* C \cdot K_X) - \dim X = -n^2(f_*(C) \cdot K_X) - \dim X > 0$$

(3) $g(C) \geq 2$. (no endomorphisms of pos degree).

Assume X and C are defined over $\mathbb{Z}$.

$$\begin{tikzcd} X \arrow[d] \\ \mathrm{Spec}(\mathbb{Z}) \end{tikzcd}$$

X_p and C_p reduction to $\overline{\mathbb{F}}_p$.

$$(y_0, \dots, y_m) \xrightarrow{F_p} (y_0^p, \dots, y_m^p)$$

inj endomorphism set-th but is a morphism of degree p.

By generic flatness, $(f_p)_*(C_p)\cdot K_{X_p}$, $g(C_p)$, $\chi(T_X|_{C_p})$ for almost all p are the same

$$C_p \xrightarrow{F_p^m} C_p \xrightarrow{f_p} X_p.$$

deform space has dim $-p^m((f_p)_*(C_p)\cdot K_{X_p}) - g(C_p)\cdot \dim(X) \geq 0$

We produce a rational curve A_p on X_p through the point.

If $A_p \cdot (-K_{X_p}) > \dim X + 1$, then A_p deforms with two fixed pts

by B&B II, $A_p \sim_{alg} A_p' + A_p''$, so that A_p' and A_p''

are rat, pass through the point and have less "degree".

In X_p, we have the curve C_p through the pt with $-(K_{X_p}).C_p \leq \dim X + 1$.

Principle: If a homogeneous system of alg eqs with coeff on $\mathbb{Z}$

has non-trivial sols over $\overline{\mathbb{F}_p}$ for ∞ many p's, then it has

a solution over any alg closed field

Idea: $Z \subseteq \mathbb{P}^N_{\operatorname{Spec} R}$, $\pi : \mathbb{P}^N_{\operatorname{Spec} R} \longrightarrow \operatorname{Spec} R$. proper.

$\pi(Z)$ is closed. If $\pi(Z)$ contains a Zariski dense set,

we have that $\pi(Z) = \operatorname{Spec} R$

Theorem: X smooth proj variety and H ample on X.

Assume there exists $C' \subseteq X$ s.t. $-(C' \cdot K_X) > 0$.

Then there exists E rational such that.

i) $\dim X + 1 \geq -(E \cdot K_X) \geq 0$

ii) $$\frac{-(E \cdot K_X)}{E \cdot H} \geq \frac{-C' \cdot K_X}{C' \cdot H}.$$

K_X is not nef

$\Downarrow$

there are rat curves

Theorem (Cone Theorem): X smooth proj.

There exists countably many curves $C_i \subseteq X$:

i) $0 < -K_X \cdot C_i \leq \dim X + 1$.

and

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum_i \mathbb{R}_{\geq 0}[C_i].$$

K_X-neg part of the cone of curves is locally polyhedral away from $K_X^{\perp}$.

Proof: Choose C_i (countable) with $0 < -(C_i \cdot K_X) \le \dim X + 1$.

$$W = \text{closure}\left(\overline{NE}_{K \ge 0} + \sum_i^{\prime} \mathbb{R}_{\ge 0}[C_i]\right)$$

$$\overline{NE}(X) \underset{\neq}{\supseteq} W$$

D positive on $W \setminus \{0\}$ and neg somewhere on $\overline{NE}(X)$.

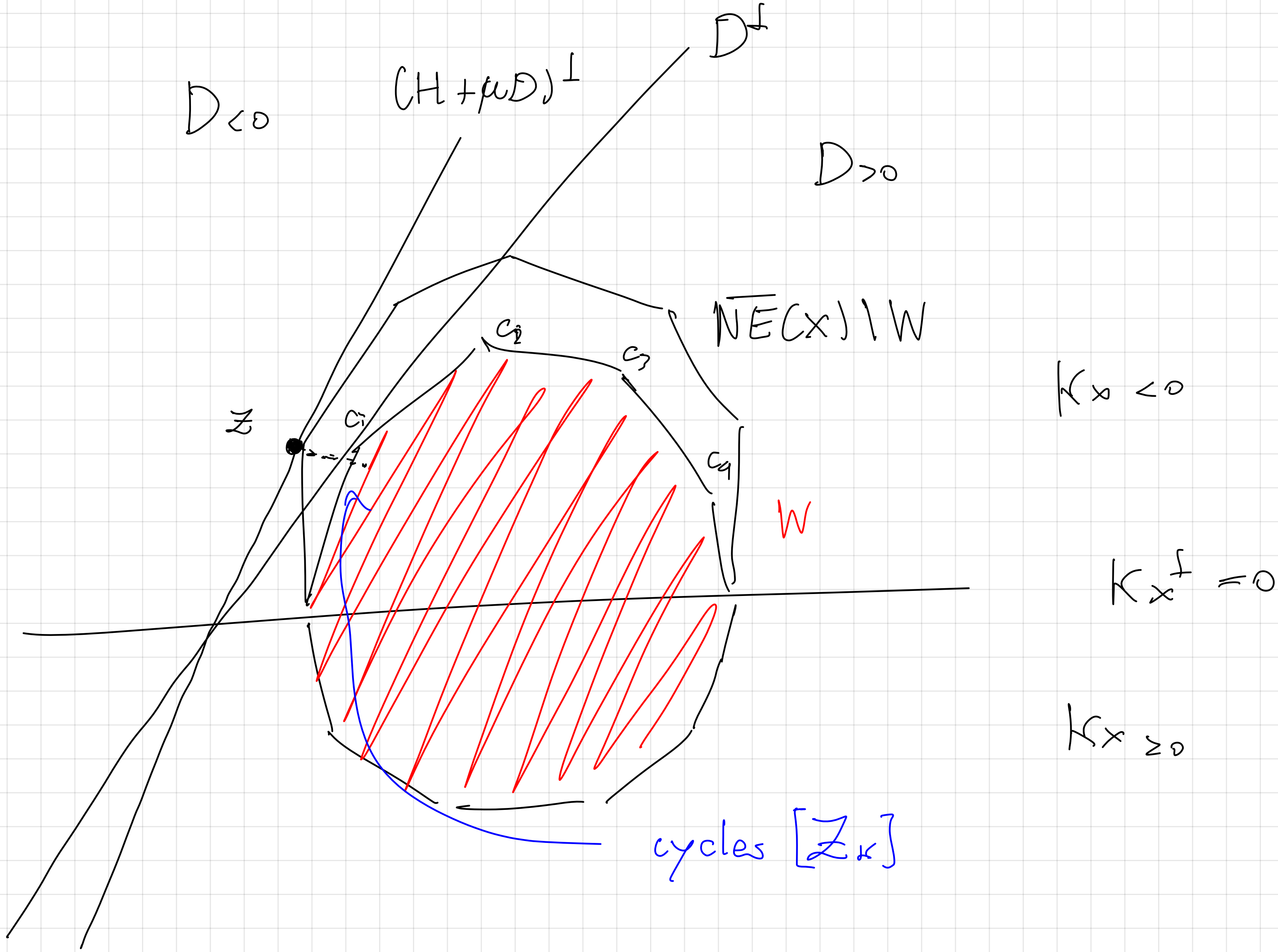
H ample. $\mu = \max\{\mu' \mid H + \mu' D \text{ is nef}\}$.

$H + \mu D$ is nef, $H + \mu' D$ is ample for $\mu' < \mu$.

$0 \neq z \in \overline{NE}(X)$. $(H + \mu D) \cdot z = 0$.

Then $K_X \cdot z < 0$, since $\overline{NE}_{K_X \ge 0} \subseteq W$.

$z_k = \sum_j^{\prime} a_{kj} z_{kj}$, $[z_k] \longrightarrow z$.



$$\max_j \frac{-(Z_{k_j}\cdot K_X)}{(Z_{k_j}\cdot(H+\mu D))} \geq \frac{-Z_k\cdot K_X}{(Z_k\cdot(H+\mu D))}$$

max attained by Z_{k_0}.

Replace Z_K with $K_X \cdot Z_K < 0$

$E_{i(K)}$ rational with

by existence of rat curves when K_X is not nef

1) $\dim X + 1 \geq -E_{i(K)} \cdot K_X > 0$.

2) $\dfrac{-E_{i(K)} \cdot K_X}{E_{i(K)} \cdot (H + \mu' D)} \geq \dfrac{-Z_{K_0} \cdot K_X}{Z_{K_0} \cdot (H + \mu' D)} \geq \dfrac{-Z_K \cdot K_X}{Z_K \cdot (H + \mu' D)}$.

by max of Z_{K_0}.

because $E_{i(K)} \cdot D \geq 0$, we have

$$\frac{-E_{i(K)} \cdot K_X}{E_{i(K)} \cdot H} \geq \frac{-Z_K \cdot K_X}{(Z_K \cdot (H + \mu' D))}$$

Fix $M \gg 0$ such that $MH + K_X$ ample.

$(MH + K_X) \cdot E_{i(K)} > 0$

$$\mu > \frac{-E_{i(\kappa)} \cdot K_X}{E_{i(\kappa)} \cdot H} \geq \frac{-Z_\kappa \cdot K_X}{Z_\kappa \cdot (H + \mu' D)}.$$

Take $\kappa \to \infty$.

$\mu' \longrightarrow \mu$.

$$\mu > \frac{Z \cdot K_X}{Z \cdot (H + \mu D)} \longrightarrow \infty.$$

$\neq 0$

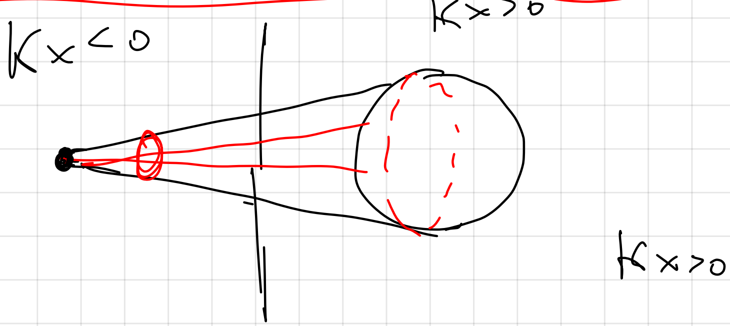
0

$\longrightarrow \longleftarrow$

K_X -neg.

arbitrary

$$\boxed{\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum_{i \text{ countable}} \mathbb{R}_{\geq 0}[C_i]}$$



$X \longrightarrow A$

$X \hookrightarrow J(X) \longrightarrow A$.

$A = E \times E$

$\overline{NE}(A)$ is rounded

Existence of rat curves.

K_X no neg $\Longrightarrow$ no rat curves.